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Chebyshev Approximation by Reciprocals of Polynomials on $[0, \infty)$

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1. INTRODUCTION

Let $C_0[0, \infty)$ denote the class of all continuous real valued functions defined on $[0, \infty)$ that vanish at ∞. Let $\|f\| = \sup_{x\in(0,\infty)} |f(x)|$ and let π_n denote the class of all algebraic polynomials of degree $\leqslant n$. Fix $B(x) \in C[0,\infty)$, where B has at most a countable number of nonnegative zeroes $\{t_\nu\}$, this set having no finite cluster point. Also, assume that there exists a positive integer N (assume that N is minimal) for which $\lim_{x\to\infty} (B(x)/X^N) = 0$. Set

$$D_0[0, \infty) = \{f\in C_0[0, \infty): f = B \cdot g, \text{ with } g \in C[0, \infty) \text{ and } g(x) > 0, \text{ for all } x \geqslant 0\} \tag{1}$$

and

$$B(n, k) = \Big\{\alpha B(x)/p^k(x): p \in \pi_n,\ n \geqslant 1,\ k \text{ a positive integer},\ n \cdot k \geqslant N,\ p(x) = \sum_{i=0}^{n} a_i x^i > 0 \text{ for } x \in [0, \infty) \text{ with } \sum_{i=0}^{n} a_i^2 = 1 \text{ and } \alpha \text{ real}\Big\}. \tag{2}$$

We write $(\alpha, p) \in B(n, k)$ whenever we are speaking about such a function in $B(n, k)$.

Functions in $D_0[0, \infty)$ are called oscillating decay-type functions and occur often in various branches of physics and chemistry. $B(x)$ is the

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"oscillation" factor and, in practice, it is often taken to be a polynomial. However, in the theory presented here, more general functions such as $B(x) = \sin x$ are also permissible.

In this paper, we study questions of best approximation of functions in $D_0[0, \infty)$ by $B(n, k)$. Thus, as usual, we call $(\alpha, p) \in B(n, k)$ a best approximation to $f \in D_0[0, \infty)$ provided that $\| f - \alpha B/p^k \| = \inf\{\| f - \beta B/q^k \| : (\beta, q) \in B(n, k)\}$. The motivation for this study comes from two sources. First, in [5], Mainardus, Reddy, Taylor, and Varga developed a Bernstein-type theory for this type of problem for the special case $B(x) = 1$ and $f \in C_0[0, \infty)$ is the reciprocal of an entire function. However, no results concerning existence, characterization, and uniqueness were given in this study. In [8], Williams studied this problem for the special case of a finite interval; existence, characterization, and uniqueness results were given, as well as a modified multiple Remes exchange for calculating best approximations. However, the existence claim was incorrect and a deeper study of this particular question was given by Taylor and Williams [7].

The main result of our study is that the standard "alternation" theorem is only sufficient. Indeed, we show that there are two types of alternation that can occur, one of which is the standard alternation. Aside from this, we show that existence is answered here as in [7] and that uniqueness holds whenever best approximations exist.

Finally, we would like to contrast this study with some results of Achieser [1]. In Chapter II of this book, the problem of finding best rational approximations in $R_n^n(-\infty, \infty)$ to functions $f \in C(-\infty, \infty)$ satisfying $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x)$ (limit being finite) is studied. In this setting, the points at $\pm\infty$ are identified and a theory of best approximation including existence, alternation (standard form, with possibility that ∞ may be an extreme point), and uniqueness are proved. Thus, in the theory given there, the results for a finite interval are essentially identical to the results for $(-\infty, \infty)$.

2. Main Results

We begin this section by proving an existence theorem similar to one given in [7].

THEOREM 1. *Assume that the zeroes* $\{t_\nu\} \subset [0, \infty)$ *of* $B(x)$ *satisfy*:

(i) $0 \leqslant t_\nu < t_{\nu+1}$, *for all* $\nu \geqslant 1$;

(ii) $\lim_{x\to 0+} | B(x)/x^k | = +\infty$, *if* $t_1 = 0$;

(iii) $\lim_{x\to t_\nu} | B(x)/(x - t_\nu)^{2k} | = +\infty$, *if* $t_\nu > 0$.

Then, for all $f \in D_0[0, \infty)$, *there exist best approximations in* $B(n, k)$.

Proof. Set $E = \text{dist}(f, B(n, k))$ and assume that $f \notin B(n, k)$. Since $\lim_{x\to\infty} B(x)/x^{nk} = 0$, we can select $\alpha > 0$ so that $\| \alpha \cdot B(x)/r^k(x) \| \leqslant (1/2) \|f\|$, where $r(x) = (1/2^{1/2})(x^n + 1)$. This inequality and the fact that $\text{sgn}(\alpha B(x)/r^k(x)) = \text{sgn} f(x)$ imply that $\|f - \alpha B(x)/r^k(x)\| < \|f\|$. Now, select ρ, $0 < \rho < 1$ such that $\|f - \alpha \cdot B(x)/r^k(x)\| = \rho \|f\|$. Thus, $E \leqslant \rho \|f\|$ and we see that 0 is not a best approximation to f. Let $\{(\alpha_m, p_m)\}_{m=1}^{\infty} \subset B(n, k)$ satisfy

$$\|f - \alpha_m B(x)/p_m{}^k(x)\| \leqslant \rho \|f\|, \qquad m = 1, 2,... \tag{3}$$

and

$$\lim_{m\to\infty} \|f - \alpha_m B(x)/p_m{}^k(x)\| = E. \tag{4}$$

Since $p_m(x) > 0$ for $x \geqslant 0$ by assumption, inequality (3) implies that $\alpha_m > 0$. Let $x_0 \in [0, \infty)$ be such that $|f(x_0)| = \|f\|$, then, (3) implies that

$$|\alpha_m| \leqslant \frac{(1 + \rho)\|f\| (\sum_{i=0}^{n} x_0^{2i})^{k/2}}{|B(x_0)|}, \qquad m = 1, 2,...,$$

where we have applied Cauchy's inequality to $|p_m(x_0)|$. Thus, we may select a subsequence $\{(\alpha_\nu, p_\nu)\}_{\nu=1}^{\infty}$ of $\{(\alpha_m, p_m)\}_{m=1}^{\infty}$ for which $\alpha_\nu \to \alpha \geqslant 0$, $p_\nu \to p \in \pi_n$, $p(x) = \sum_{i=0}^{n} a_i x^i$, with $\sum_{i=0}^{n} a_i{}^2 = 1$ and the convergence of p_ν to p is uniform on each compact subset of $[0, \infty)$. Now, at $\tilde{x}_0$, where $\tilde{x}_0$ is chosen so that $|f(\tilde{x}_0)| > \rho \|f\|$ and $p(\tilde{x}_0) \neq 0$, we have that $|f(\tilde{x}_0) - \alpha B(\tilde{x}_0)/p^k(\tilde{x}_0)| = \lim_{\nu\to\infty} |f(\tilde{x}_0) - \alpha_\nu B(\tilde{x}_0)/p_\nu{}^k(\tilde{x}_0)| \leqslant E < \rho \|f\|$, implying that $\alpha > 0$. Fix $x \in [0, \infty)$, then,

$$\left| f(x) - \frac{\alpha B(x)}{p^k(x)} \right| \leqslant \left| f - \frac{\alpha_\nu B(x)}{p_\nu{}^k(x)} \right| + \left| \frac{\alpha_\nu B(x)}{p_\nu{}^k(x)} - \frac{\alpha B(x)}{p^k(x)} \right|.$$

Letting $\nu \to \infty$ and then taking the sup over all $x \in [0, \infty)$, gives

$$\|f - \alpha B/p^k\| \leqslant E. \tag{5}$$

Finally, we must show that $p(x) > 0$ for all $x \geqslant 0$ so that $(\alpha, p) \in B(n, k)$. Now, (3) implies that $p(x) > 0$, whenever $B(x) \neq 0$. Thus, we must only show that $p(t_\nu) > 0$ for all $\nu \geqslant 1$. Now, suppose that $t_1 = 0$ and $p(0) = 0$. Then, by (ii) we would have that $\lim_{x\to 0+} |\alpha B(x)/p^k(x)| = +\infty$ violating (5). Thus, $p(0) > 0$. Finally, suppose that $p(t_\nu) = 0$ for some zero $t_\nu > 0$ of $B(x)$. Since $p(x) > 0$ for all $x \notin \{t_\nu\}$ and $\{t_\nu\}$ has no finite cluster point, we must have that t_ν is an even-order zero of multiplicity $\geqslant 2$. But this would then violate (5) because of assumption (iii). Hence, $p(x) > 0$ for $x \geqslant 0$ and thus, $(\alpha, p) \in B(n, k)$ with

$$\|f - \alpha B/p^k\| = E.$$

Observe that for the best approximation $(\alpha, p) \in B(n, k)$, it may be possible that $\overline{\lim}_{x\to\infty} \sup | \alpha \cdot B(x)/p^k(x)| = E$. Also, if $B(x) \equiv 1$, then Theorem 1 guarantees the existence of best approximations to positive functions in $C_0[0, \infty)$ by elements of the form $1/p$, $p \in \pi_n$, $p > 0$ on $[0, \infty)$.

Next, we wish to turn to the problem of characterizing best approximations. As usual, for a given $f \in D_0[0, \infty)$ and $(\alpha, p) \in B(n, k)$ we say that $x \in [0, \infty)$ is an extreme point for f (with respect to (α, p)) provided that $|f(x) - \alpha B(x)/p^k(x)| = \|f - (\alpha B/p^k)\|$. Before proving our alternation theorem, we wish to give a simple example that shows that the standard alternation theorem is not a necessary condition for best approximations in this setting. Set $B(x) \equiv 1$, $k = 1$, $n = 2$, and $p(x) = x + 1$. Dfine $f \in C_0[0, \infty)$ by $f(0) = 5/4, f(1) = 1/4, f(2) = 7/12$, f is linear on $[0, 2]$ and $f(x) = 7/12\, e^{-(x-2)}$ for $x \geqslant 2$. Note that the points $x = 0, 1, 2$ are extreme points for f and $\|f - 1/(x + 1)\| = 1/4$. If there exists $q(x) = ax^2 + bx + c \in B(2, 1)$ such that $\|f - 1/q\| \leqslant \|f - 1/(x + 1)\|$, then we must have $2/3 \leqslant c \leqslant 1, (x = 0)$; $a + b + c \geqslant 2, (x = 1)$; and $6/5 \leqslant 4a + 2b + c \leqslant 3\ (x = 2)$ and $a \geqslant 0$. The only solution to this system of inequalities is $q(x) = x + 1$. Hence, $x + 1$ is the unique best approximation to f from $B(2, 1) \equiv R_0^2[0, \infty)$. Yet, the standard theory for $R_2^0[0, N]$ $N > 0$, requires the best approximation to alternate on a set of at least four extreme points. Thus, the standard alternation theorem is not a necessary condition in this case.

We now turn to proving our characterization theorem. For convenience, we change our normalization of elements of $B(n, p)$ by writing α/p as $1/q$, where $q = p/\alpha$ whenever $\alpha \neq 0$. Recall that one consequence of our existence theorem was that 0 is not a best approximation to any function in $D_0[0, \infty) \sim B(n, k)$. Thus, in what follows, we consider only nonzero elements of $B(n, k)$. Also, we would like to point out that one could apply the asymptotic convexity theory of Meinardus and Schwedt [6] to this problem to arrive at the same result. However, we prefer to derive this result from first principles.

THEOREM 2. *Let $f \in D_0[0, \infty) \sim B(n, k), f = B \cdot g$ and $p \in B(n, k)$, $p > 0$ on $[0, \infty)$. Then, p is a best approximation to f if and only if one the following two conditions hold:*

(a) *There exist points $0 \leqslant x_1 < x_2 < \cdots < x_{n+2}$ at which $|f(x_i) - B(x_i)/p^k(x_i)| = \|f - B/p^k\|$, $i = 1,\ldots, n + 2$ and $\operatorname{sgn}(g(x_i) - 1/p^k(x_i)) = -\operatorname{sgn}(g(x_{i+1}) - 1/p^k(x_{i+1}))$, $i = 1,\ldots, n + 1$.*

(b) *The degree of $p \leqslant n - 1$ and there exist points $0 \leqslant x_1 < \cdots < x_{n+1}$ at which $|f(x_i) - B(x_i)/p^k(x_i)| = \|f - B/p^k\|$, $i = 1,\ldots, n + 1$ and $\operatorname{sgn}(g(x_i) - 1/p^k(x_i)) = (-1)^{n+1-i}$.*

Proof. Set $e(x) = f(x) - B(x)/p^k(x)$ and $E = \|e(x)\|$. Suppose that $p \in B(n, k)$ is a best approximation to f and that (a) does not hold. In this event, we wish to prove that (b) must hold. We do this by deriving a series of contradictions. However, first note that the error curve $e(x) = f(x) - B(x)/p^k(x)$ cannot be a constant since $p(x) + \epsilon$ is in $B(n, k)$ for sufficiently small $\epsilon > 0$. Now, let us suppose that there exists $m \leqslant n$ points, $x_1, \ldots, x_m$, on which $|e(x_i)| = E$, $i = 1,\ldots, m$ and

$$\operatorname{sgn}(g(x_i) - 1/p^k(x_i)) = -\operatorname{sgn}(g(x_{i+1}) - 1/p^k(x_{i+1})),\ i = 1,\ldots, m-1$$

hold with m maximal. Following the format of the corresponding argument in the classical problem of approximating with polynomials on a finite interval [4, p. 26–27], we select points $\{z_i\}_{i=1}^{m-1}$ such that $x_i < z_i < x_{i+1}$, $i = 1,\ldots, m-1$, $e(z_i) = 0$, $i = 1,\ldots, m-1$ and there are no alternations as defined in (b) in each of the intervals $[z_i, z_{i+1}]$, $i = 0,\ldots, m-1$, where $z_0 = 0$ and $z_m \geqslant x_m + 1$ such that $|f(x)| \leqslant E/2$ for all $x \geqslant z_m$.

Set $\varphi(x) = \prod_{i=1}^{m-1} (x - z_i)$ and note that $\partial\varphi \leqslant n - 1$. Now, modify p according to the following two cases. (i) Suppose that $\operatorname{sgn}(e(x_m)/B(x_m)) = -1$. In this case, set $q(x) = p(x) + \epsilon(x + 1)^{n-m+1}\varphi(x)$, where $\epsilon > 0$ is chosen so that

$$\max_{x\in[0,z_m]} |f(x) - B(x)/q^k(x)| < E.$$

That such a choice for ϵ exists is easily seen using essentially the compactness and continuity arguments for the corresponding argument in the classical polynomial approximation problem on a finite interval [4, p. 26–27]. Now, for $x \geqslant z_m$ and observe that $1/q(x) < 1/p(x)$ and $\lim_{x\to\infty} |B(x)/q^k(x)| = 0$. Thus, there exists $M \geqslant z_m$ such that $x \geqslant M$ implies

$$|f(x) - B(x)/q^k(x)| < E/2.$$

Also, on the interval $[z_m, M]$ we have that whenever $B(x) \neq 0$, $|f(x) - B(x)/q^k(x)| < E$. Thus, by continuity,

$$\max_{x\in[z_m,M]} |f(x) - B(x)/q^k(x)| < E.$$

Combining these results gives that

$$\|f - B/q^k\| < E,$$

which is a contradiction.

Now, consider (ii) $\operatorname{sgn} e(x_m)/B(x_m) = +1$. In this case, set $q(x) = p(x) + \epsilon(x + 1)^{n-m}\varphi(x)(x - z_m)$, where $(x + 1)^{n-m}$ is to be replaced by 1 if $n = m$ and $\epsilon > 0$ is chosen so that

$$\max_{x\in[0,z_m]} |f(x) - B(x)/q^k(x)| < E.$$

That it is possible to choose an $\epsilon > 0$ satisfying this requirement follows readily from the fact that on (z_i, z_{i+1}) $i = 0,..., m - 1$, $\text{sgn}((x + 1)^{n-m} \varphi(x)(x - z_m)) = -\text{sgn } e(x_i)/B(x_i)$. Now, for $x \geqslant z_m$, we have by our assumption that m is maximal, $|e(x)| < E$, although it could happen that $\overline{\lim}_{x\to\infty} |e(x)| = E$. However, as in case (i), we have that $\lim_{x\to\infty} B(x)/q^k(x)=0$ and for $x \geqslant z_m$, $1/q(x) \leqslant 1/p(x)$. Thus, once again

$$\|f - B/q^k\| < E,$$

contradicting our hypothesis that p is a best approximation to f from $B(n, k)$. Thus, we must have at least $n + 1$ extremals, $0 \leqslant x_1 < \cdots < x_{n+1}$ for which $|e(x_i)| = E$ $i = 1,..., n + 1$ and $\text{sgn}(g(x_i) - 1/p^k(x_i)) = -\text{sgn}(g(x_{i+1}) - 1/p^k(x_{i+1}))$, $i = 1,..., n$. Since we are assuming that (a) does not hold, we must have that there are exactly $n + 1$ such points.

Now, let us show that $\partial p \leqslant n - 1$ also must be satisfied. Indeed, if $\partial p = n$, then, since $p > 0$ for all $x \in [0, \infty)$ we must have that the leading coefficient of p is positive. Now, let $m \geqslant x_{n+1}$ be such that $x \geqslant M$ implies $|e(x)| \leqslant E/2$. Such an M exists since $\lim_{x\to\infty} |e(x)| = 0$ as $\partial p = n$. Select $\{z_i\}_{i=0}^{n+1}$ such that $z_0 = 0$, $z_{n+1} = M$, $x_i < z_i < x_{i+1}$, $i = 1,..., n$, $e(z_i) = 0$, $i = 1,..., n$ and there are no alternations as defined in (b) in $[z_i, z_{i+1}]$, $i = 0,..., n$. Set $\varphi(x) = \prod_{i=1}^{n} (x - z_i)$ and let $\epsilon_0 > 0$ be chosen so that $p(x) - \epsilon_0 |\varphi(x)| > 0$ for $x \geqslant 0$. Such an ϵ_0 can be shown to exist since $\partial p = n$. Now, set $q(x) = p(x) + \epsilon\varphi(x)$, where $|\epsilon| \leqslant \epsilon_0$, is chosen so that $q(x)$ is a better approximation than $p(x)$ to f. Indeed, $\text{sgn } \epsilon = -\text{sgn } e(x_{n+1})/B(x_{n+1})$ and $|\epsilon|$ sufficiently small will guarantee that

$$\max_{x\in[0,z_{m+1}]} |f(x) - B(x)/q^k(x)| < E,$$

by the standard argument cited above. Next, select N such that $N \geqslant z_{n+1}$ and $x \geqslant N$ implies $|B(x)/(p(x) - |\epsilon| \varphi(x))^k| \leqslant E/3$ and $|f(x)| \leqslant E/3$. Then, by possibly making $|\epsilon|$ smaller, we can guarantee that

$$\max_{x\in[z_{n+1},N]} |f(x) - B(x)/q^k(x)| < E$$

and for $x \geqslant N$,

$$|f(x) - B(x)/q^k(x)| \leqslant 2E/3.$$

Thus, $\|f - B/q^k\| < E$ and we have our desired contradiction, so that $\partial p \leqslant n - 1$ must hold.

Finally, we wish to show that $\text{sgn}(g(x_{n+1}) - 1/p^k(x_{n+1})) = 1$. If not, then the argument of case (i) above can be repeated to get our desired contradiction. Thus, we have proved the necessity part of the theorem.

Now, let us turn to proving that both (a) and (b) are sufficient conditions for p to be the best approximation to f from $B(n, k)$. Thus, assume that condition (a) is satisfied by p. In this case, p is not only the *unique* best approximation to f from $B(n, k)$; but in fact, p is the unique best approximation to f from $B(n, k)$ on the closed interval $[0, x_{n+2}]$ by the results of Williams [8].

Thus, suppose that condition (b) holds and that there exists $q \in B(n, k)$ for which

$$\|f - B/q^k\| \leqslant \|f - B/p^k\|.$$

(We shall actually prove that p is unique whenever (b) holds in our argument.) Thus, at the extreme points $x_1, ..., x_{n+1}$, we have that

$$(-1)^{n+1-i} (p(x_i) - q(x_i)) \geqslant 0, \qquad i = 1, ..., n + 1.$$

Using the standard zero counting argument (for uniqueness) [1, p. 56–57], we have that either $\partial q = n$ or $q \equiv p$ as $\partial p \leqslant n - 1$. Thus, suppose that $\partial q = n$. Since $\|f - B/q^k\| \leqslant E$, we must have that $q(x) > 0$ for all $x \geqslant 0$, so that the leading coefficient of q is positive. Hence, there exists $x_{n+2} > x_{n+1}$ at which $q(x_{n+2}) - p(x_{n+2}) > 0$. Adjoining x_{n+2} to the set $x_1, ..., x_{n+1}$, we have that

$$(-1)^{n+1-i} (p(x_i) - q(x_i)) \geqslant 0, \qquad i = 1, ..., n + 2.$$

Once again, appealing to [1, p. 56–57], gives $p \equiv q$. ∎

COROLLARY 1. *Best approximations from $B(n, k)$ to functions of $D_0[0, \infty)$ are unique.*

Thus, we see that best approximations are characterized by two possible types of alternation. It should be noted that condition (a) of Theorem 2 corresponds to the case when the approximation problem is equivalent to approximating the given function on some interval of the form $[0, N]$, N sufficiently large. Also, Theorem 2 implies that for a given $f \in D_0[0, \infty)$, one cannot expect the Remes algorithm to necessarily converge to a best approximation. This is so, since the Remes algorithm, without some sort of modification, will always find the best approximation on some interval of the form $[0, N]$, $N > 0$. However, if one is fortunate in his choice of $f(x)$, such as $f(x) = e^{-x}$ as used in [3], then the Remes algorithm will give the desired answer for N sufficiently large.

In [2], theorems corresponding to a zero in the convex hull characterization, strong uniqueness and continuity of the best approximation operator are studied for the special case $B(x) \equiv 1$. For example, if a best approximation $p^* \in B(n, k)$ satisfies $\partial p^* < n - 1$, then the best approximation operator need not be continuous at p^*, for $\partial p^* = n$, the operation is continuous at p^* and for $\partial p^* = n - 1$, this question is open.

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